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An Improved Algorithm for Computing Topological Entropy

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A new algorithm is presented for computing the topological entropy of a unimodal map of the interval. The accuracy of the algorithm is discussed and some graphs of the topological entropy which are obtained using the algorithm are displayed.

KEY WORDS: Topological entropy; unimodal map; chaotic dynamical system; kneading sequence.

INTRODUCTION

Iterated maps of the interval, when viewed as dynamical systems, have become important models for the chaotic behavior observed in certain physical, chemical, and biological systems. The topological entropy of a map is one of the indicators of the complexity of the system. Collet *et al.*[3] gave an algorithm for computing the topological entropy of a unimodal map. In this paper we present a new algorithm, which appears to give more accurate results when actually implemented on a computer. We also discuss the accuracy of the new algorithm and present graphs of the topological entropy versus the parameter for some one-parameter families of maps. The values of the topological entropy used in plotting the graphs are obtained using the algorithm.

1. PRELIMINARIES

The topological entropy of a map, denoted by $h(f)$, was first defined by Adler *et al.*[1] Alternate definitions (which are equivalent to the original)

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were given by Bowen.[2] Several theoretical results could also be used to give alternate definitions. For example, for a piecewise-monotone map f the topological entropy is the exponential growth rate of the number of monotone pieces of the graph of the nth iterate of f and also the exponential growth rate of the total variation of the nth iterate of f.[8] However, except in very special cases, the entropy cannot be easily computed using these definitions and results. Thus, the problem of finding an algorithm to compute topological entropy is not immediately solved by the theoretical results.

Both the paper of Collet *et al.*[3] and this paper deal with the special case of unimodal maps. We say a map f defined on a closed interval $[a, b]$ is *unimodal* if f is continuous on $[a, b]$ and there is a number c in the open interval (a, b) such that f is strictly increasing on $[a, c]$ and strictly decreasing on $[c, b]$. Of course, everything in the sequel could be easily modified to cover the case where f is strictly decreasing on $[a, c]$ and strictly increasing on $[c, b]$.

Let f be a unimodal map. Let f^n denote the nth iterate of f, i.e., f^n is the composition of f with itself n times. The *kneading sequence* of f is the sequence

$$K(f) = K_1 K_2 K_3 \ldots$$

defined by

$$K_i = \begin{cases} R & \text{if } f^i(c) > c \\ C & \text{if } f^i(c) = c \\ L & \text{if } f^i(c) < c \end{cases}$$

The algorithm given in ref. 3 uses a formula for the topological entropy of f, based on the smallest positive root of a certain power series whose coefficients are obtained from the kneading sequence. We present here a different algorithm which also uses the kneading sequence, but appears to give more accurate results with fewer terms of the kneading sequence. The algorithm also avoids the numerical problems inherent in finding roots of high-degree polynomials. The main idea of our algorithm is to use a model family of maps whose entropy is known, and to compare the kneading sequence for the given map f (whose entropy we wish to compute) with the kneading sequences of the model family.

For the model family we use the family of unimodal maps known as "tent maps" defined for parameters with $0 \leqslant s \leqslant 2$ on the interval $[0, 1]$ by

$$f_s(x) = \begin{cases} sx & \text{if } 0 \leqslant x \leqslant 1/2 \\ s - sx & \text{if } 1/2 \leqslant x \leqslant 1 \end{cases}$$

The topological entropy of f_s is well known.

Proposition 1.1:

$$h(f_s) = \begin{cases} 0 & \text{if } 0 \leqslant s \leqslant 1 \\ \log s & \text{if } 1 \leqslant s \leqslant 2 \end{cases}$$

This proposition follows, for example, from the formula of Misiurewicz and Szlenk,[8] which is valid for piecewise monotone maps f of the interval,

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log \operatorname{Var} f^n$$

where Var denotes the total variation.

To compare kneading sequences, we use the standard ordering given as follows. First we order the symbols R, C, and L by $L < C < R$. Now, if $A = A_1 A_2 A_3 \cdots$ and $B = B_1 B_2 B_3 \cdots$ are kneading sequences, we say $A < B$ if there is an s such that $A_i = B_i$ for $i = 1, 2, \dots, s$ and either (1) an even number of A_i, $i \leqslant s$, are equal to R and $A_{s+1} < B_{s+1}$, or (2) an odd number of A_i, $i \leqslant s$, are equal to R and $A_{s+1} > B_{s+1}$.

This gives a total ordering on the set of all kneading sequences. Our algorithm is based on the following characteristic of this ordering (see Lemma 2 of ref. 3).

Proposition 1.2. If f and g are unimodal maps and $K(f) \leqslant K(g)$, then $h(f) \leqslant h(g)$.

We will use the notation $K_N(f)$ to denote the first N terms of the kneading sequence of f. If $K_N(f) = A_1 A_2 \cdots A_N$ and $K_N(g) = B_1 B_2 \cdots B_N$, we say $K_N(f) < K_N(g)$ if there is an s with $1 \leqslant s < N$ such that $A_i = B_i$ for $i = 1, \dots, s$ and either (1) or (2) holds. It follows that if $K_N(f) < K_N(g)$, then $K(f) < K(g)$. Thus we obtain the following.

Proposition 1.3. If $K_N(f) < K_N(g)$ for some positive integer N, then $h(f) \leqslant h(g)$.

To simplify the notation, if $0 \leqslant s \leqslant 2$ and N is a positive integer, we let $K_N(s)$ denote $K_N(f_s)$, where f_s is the tent map with slope s defined earlier.

2. THE ALGORITHM

We now describe a theoretical algorithm which suggests itself from the above discussion. Suppose a unimodal map f and a positive number ε are given. We will compute $h(f)$ with an error at most ε.

Step 1. Let M be a positive integer such that $\delta = 1/M < \varepsilon$.

Step 2. Find the least positive integer N so that the M finite sequences $K_N(1)$, $K_N(1+\delta)$, $K_N(1+2\delta)$, $K_N(1+3\delta)$,..., $K_N(1+M\delta)=K_N(2)$ are distinct.

Step 3. Compute $K_N(f)$.

Step 4. Let R be the largest integer with $K_N(1+R\delta)<K_N(f)$.

Step 5. Set $h(f)=\log[1+(R+1)\,\delta]$.

It follows from Propositions 1.1 and 1.3 that $\log(1+R\delta)\leqslant h(f)\leqslant \log[1+(R+2)\,\delta]$. It is easy to show by the mean value theorem that the error in estimating $h(f)$ by $\log[1+(R+1)\,\delta]$ is less than $\delta<\varepsilon$.

3. PRACTICAL IMPLEMENTATION OF THE ALGORITHM

The previous section describes an algorithm that will compute the topological entropy of a unimodal map to any prescribed accuracy. There are some obstacles to implementing this algorithm in the precise form described in the previous section. Here we describe a realization of the algorithm which seems to work quite well in practice.

The main criterion to be applied to the program is accuracy. Of course, simplicity and speed are also important. The accuracy of the program described above will be the given prescribed error $\varepsilon>0$ provided that all computations are done without roundoff error. This would be untenable. Where will roundoff errors cause difficulties in the algorithm? They will occur when the function f and the functions f_s are evaluated. The accuracy of with which these functions are computed determines the accuracy of the kneading sequences. Thus, while one may need N terms in the kneading sequence to determine $h(f)$ to accuracy ε, one may only obtain some $N'<N$ terms accurately because of the error in computing f. If a standard compiler is used, the accuracy will likely be limited to double precision or roughly 14 digits. With double precision, it was empirically found that the number of terms in the kneading sequence that were accurate for f_s would be about 900 close to $s=1$ and decrease exponentially to about 50 for $s=2$. Fortunately, it also turns out that empirically the number of terms needed for a given accuracy $\varepsilon>0$ in determining $h(f)$ is larger near $s=1$ and decreases exponentially to a small number near $s=2$ as well. Some graphs are included here to help visualize these facts (Figs. 1 and 2).

There are several aspects of the theoretical algorithm which would make it unacceptably slow on a typical desktop computer. One is computing all of the kneading sequences $K_N(s)$ each time one computes the entropy for a function f. One could avoid this difficulty by choosing a given

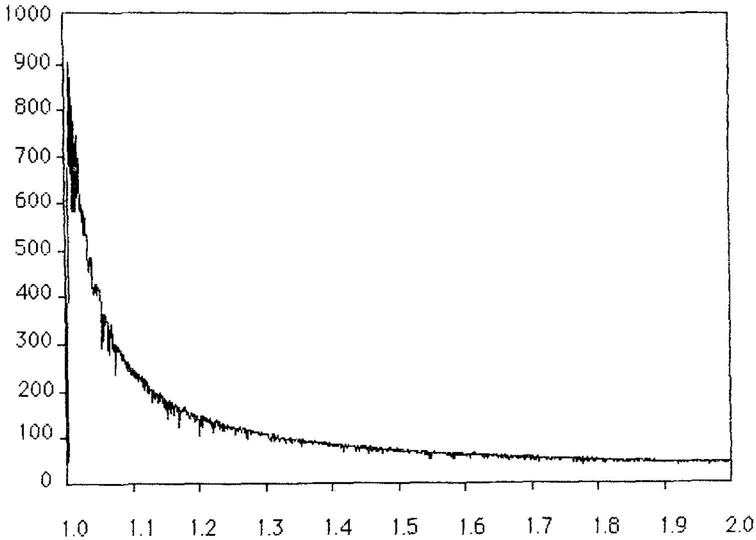

Fig. 1. The number of correct terms of the kneading sequence versus the parameter s of the tent map f_s using 13-digit precision in computation. The x axis in the figure is the parameter s, the y axis is the number of terms that can be computed in the kneading sequence before there is ambiguity as to what the next term should be. The step size for s in computing this graph is 0.001.

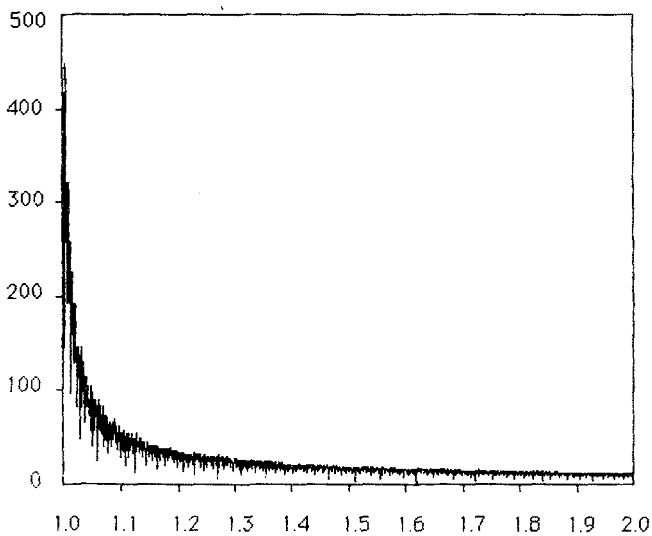

Fig. 2. The number of terms in the kneading sequence needed for $K_N(s_i)$ and $K_N(s_{i+1})$ to be distinct versus the parameter s of the tent map f_s. The x axis in the figure is the parameter s, the y axis is the number of terms N in the kneading sequence needed so that $K_N(s_i)$ and $K_N(s_{i+1})$ are distinct. The difference between s_i and s_{i+1} is 0.001.

$\varepsilon > 0$ and finding the integers N and M from Steps 1 and 2 of the algorithm. Then one could compute the $M+1$ kneading sequences in Step 2 once for all. To compute the topological entropy of any map f with error at most ε would then only require that one go through Steps 3–5 of the algorithm. The difficulty with this would be that one would not be able to let ε be smaller for greater accuracy in a subsequent application. Also, if ε were extremely small, the storage of the $M+1$ kneading sequences could be a problem.

These problems are avoided by using a method akin to the bisection method in numerical analysis. In this algorithm a positive integer N is chosen which will generate the accuracy desired. Let $s_1 = 1$ and $s_2 = 2$.

Step 1. Choose a positive integer N.

Step 2. Compute $K_N(f)$.

Step 3. Let $s = (s_1 + s_2)/2$ and compute $K_N(f_s)$.

Step 4. If $K_N(f) = K_N(f_s)$ or if $(s_2 - s_1)/2$ is less than the roundoff error of the computer, then estimate $h(f)$ by $\log s$. The error in this estimate is at most $(s_2 - s_1)/2$. If $K_N(f) \neq K_N(f_s)$ and $(s_2 - s_1)/2$ is still greater than the roundoff error of the computer, then go to Step 5.

Step 5. If $K_N(f) > K_N(f_s)$, then $\log s_2 \geqslant h(f) \geqslant \log s$. In this case let s_2 be the same and let $s_1 = s$ and go to Step 3. If $K_N(f) < K_N(f_s)$, then

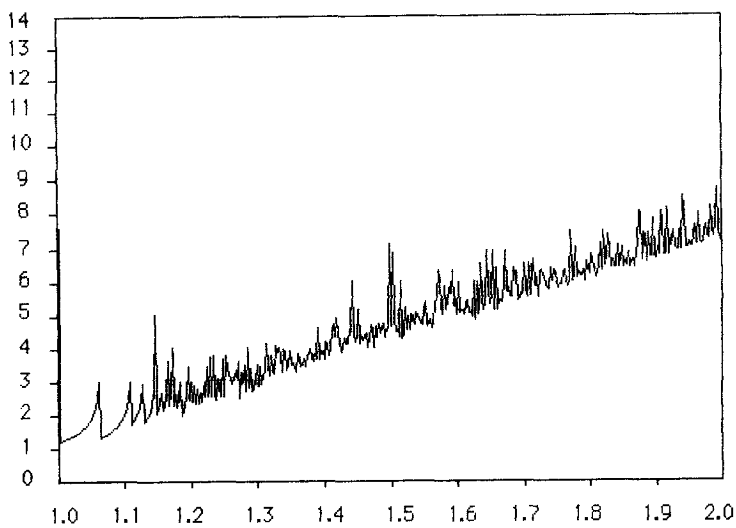

Fig. 3. Number of digits accuracy obtained using 25 terms in the kneading sequence versus the parameter s for the tent map f_s.

$\log s_1 \leqslant h(f) \leqslant \log s$. In this case let s_1 be the same and let $s_2 = s$ and go to Step 3.

One could modify the algorithm so that the number of terms in the kneading sequence varies according to the accuracy desired, but it is much simpler to hold this number fixed. In order to aid in determining how large N must be in order to achieve the desired accuracy, various graphs have been prepared (Figs. 3–6). The value s of the tent map is plotted on the x axis. The y axis is the number of digits accuracy obtained by the above algorithm for the number N. This is done for $N = 25$, 50, 100, and 200.

Note in these graphs that there is little effect in increasing the number of terms in the kneading sequence near $s = 1$. The relationship between s and the number of digits accuracy is roughly linear with the number of terms N in the sequence $K_N(s)$ determining the "slope" of this line. Two or three digits is usually enough accuracy for graphs of the topological entropy. One can easily get 6–14 digits accuracy with 50–60 terms in the kneading sequence for most of the range of the parameter. When one is near $h(f) = 0$, the number of terms needed to get even three digits accuracy is quite high. From Fig. 2 one would need approximately 450 terms for this much accuracy. Fifty terms were used to produce the graphs of the entropy in the next section of the paper.

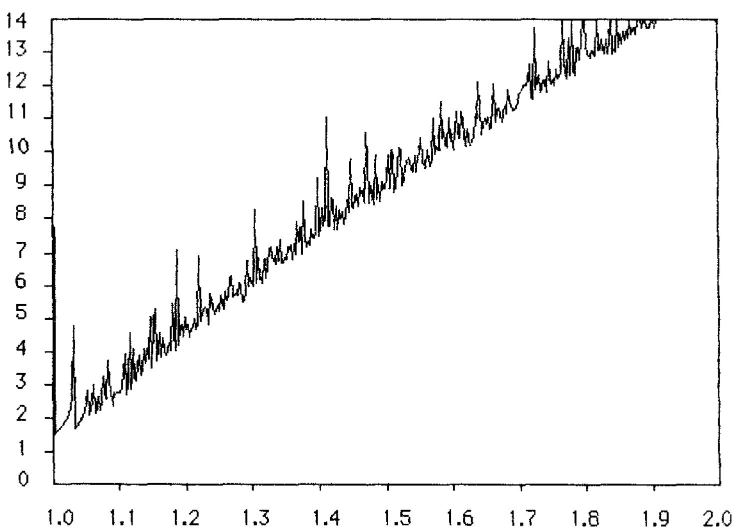

Fig. 4. Number of digits accuracy obtained using 50 terms in the kneading sequence versus the parameter s for the tent map f_s.

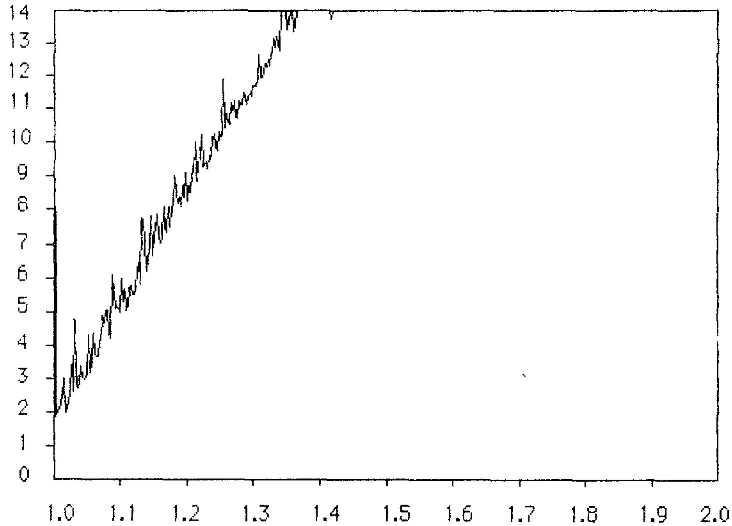

Fig. 5. Number of digits accuracy obtained using 100 terms in the kneading sequence versus the parameter s for the tent map f_s.

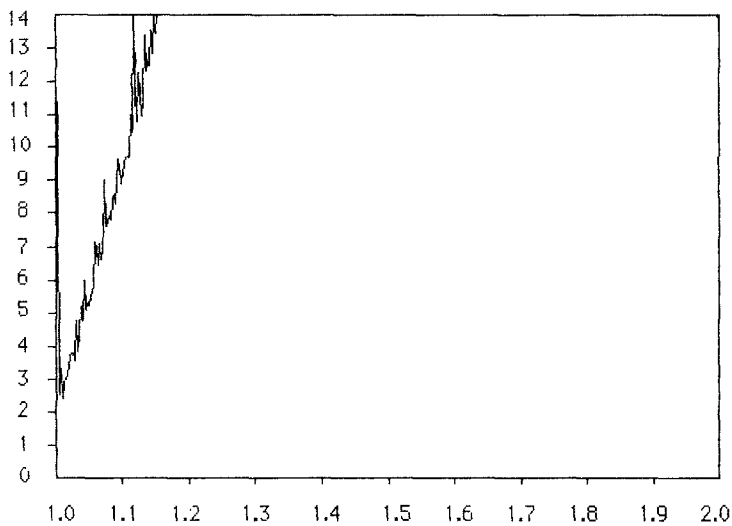

Fig. 6. Number of digits accuracy obtained using 200 terms in the kneading sequence versus the parameter s for the tent map f_s.

4. GRAPHS OF THE TOPOLOGICAL ENTROPY

Let $f_\mu(x) = \mu x(1-x)$ be the quadratic family of maps defined on the interval [0, 1]. Assume that the parameter μ is in the interval [0, 4] so that f_μ maps into [0, 1]. This family of maps is important in theoretical population dynamics and is an example of a family of simple maps with extremely complicated dynamics.[6] In this section the topological entropy of this map is graphed as a function of the parameter μ (Fig. 7). It is well known that the topological entropy is zero for μ in most of the interval [0, 4]. The plot is given only in the range [3.5, 4]. This includes all the values where the topological entropy is not zero. The plot of the bifurcation diagram of the quadratic family is also included for the same interval of μ for comparison (Fig. 8). Note that on those intervals of μ where there are attracting periodic points for the map f_μ, the topological entropy is constant. This is known for theoretical reasons to be true. That the graphs clearly indicate this gives additional confidence in the algorithm.

The algorithm described in the previous section makes the graph of the topological entropy quite easy to plot and only uses approximately 15 min of computer time with a Macintosh computer and True BASIC.

Our final graph (Fig. 9) is the growth number (i.e., the number whose logarithm is the topological entropy) versus the parameter a for the family of maps $f_a(x) = (a - x^2)/2$ mentioned in Milnor and Thurston.[7]

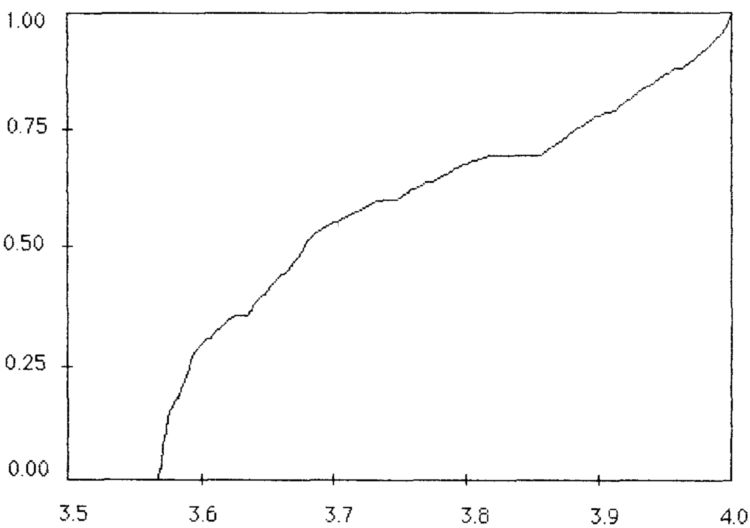

Fig. 7. Graph of the topological entropy versus the parameter μ for the quadratic family $\mu x(1-x)$. The x axis is the parameter μ of the map $f_\mu(x) = \mu x(1-x)$ for μ in the range [3.5, 4]. The y axis is the topological entropy of the map f_μ using logarithms base 2 so that the y axis is [0, 1].

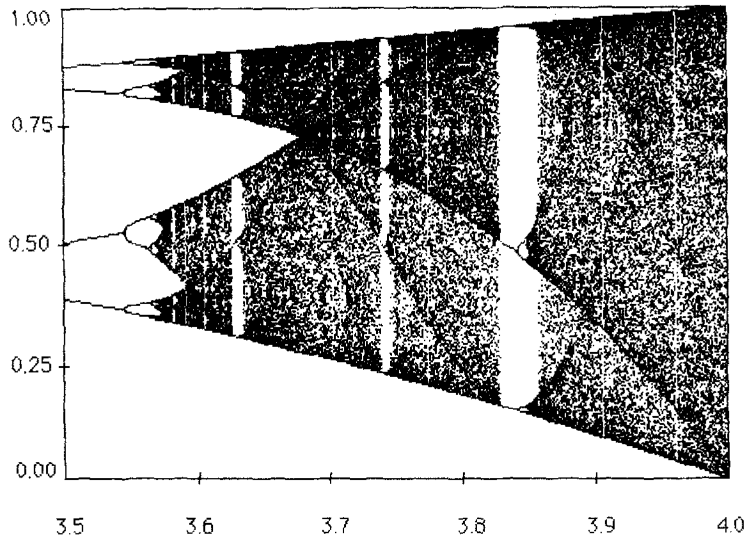

Fig. 8. Bifurcation diagram for the quadratic family of maps $\mu x(1-x)$ for $3.5 \leqslant \mu \leqslant 4$. This graph is obtained in the following manner. Determine a size for the increment $\Delta\mu = 1/2n$. Then for a fixed $0 \leqslant j \leqslant n$, let $\mu_j = 7/2 + j\,\Delta\mu$. Let $x_0 = 1/2$. Let $x_{i+1} = f_\mu(x_i)$ for $i = 1$–50. Then plot (μ_j, x_i) for $i = 51$–250. Do this for each $0 \leqslant j \leqslant n$.

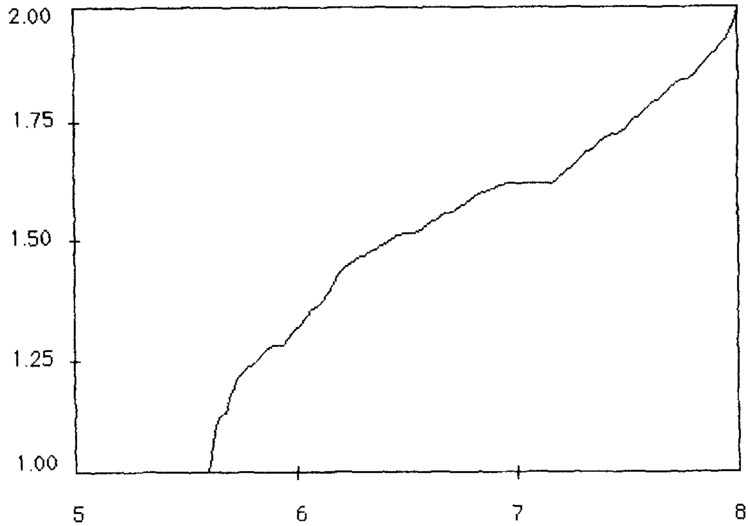

Fig. 9. Growth number versus the parameter a for the family $(a - x^2)/2$ with $5 \leqslant a \leqslant 8$.

5. CONCLUDING REMARKS

The algorithm described in this paper is useful for computing the topological entropy of unimodal maps on an interval. It is easily programmed and is fast and accurate enough to produce faithful graphs of the topological entropy of one-parameter families of unimodal maps. With double precision accuracy the algorithm works quite well for the typical unimodal maps that arise in applications.

REFERENCES

1. R. L. Adler, A. C. Konheim, and M. H. McAndrew, Topological entropy, *Trans. Am. Math. Soc.* **114**:309–319 (1965).
2. R. Bowen, Entropy for group endomorphisms and homogeneous spaces, *Trans. Am. Math. Soc.* **153**:401–414 (1971).
3. P. Collet, J. P. Crutchfield, and J.-P. Eckmann, Computing the topological entropy of maps, *Commun. Math. Phys.* **88**:257–262 (1983).
4. P. Collet and J.-P. Eckmann, *Iterated Maps of the Interval as Dynamical Systems* (Birkhäuser, Boston, 1980).
5. R. L. Devaney, *Chaotic Dynamical Systems* (Benjamin/Cummings, Menlo Park, 1986).
6. R. May, Simple mathematical models with very complicated dynamics, *Nature* **261**:459–467 (1976).
7. J. Milnor and W. Thurston, On iterated maps of the interval I, II, Preprint, Institute for Advanced Study, Princeton (1977).
8. M. Misiurewicz and W. Szlenk, Entropy of piecewise monotone mappings, *Studia Math.* **67**:45–63 (1980).